Thm (G. and Hart)
If CEP is true, then $\mathrm{Th}_\forall(\mathcal{R})$ is computable, i.e. there is an algorithm s.t. upon input universal sentence σ and $\varepsilon \in \mathbb{Q}^{>0}$, returns $I \subset \mathbb{R}$, $|I| < \varepsilon$, with $\sigma^{\mathcal{R}} \in I$.

Now: Thm (G. and Hart)
$MIP^* = RE \Rightarrow \mathrm{Th}_\forall(\mathcal{R})$ is not comp!

$p \in [0,1]^{k^2 n^2}$ is a synchronous strategy if $p(a,b|x,x) = 0$ if $a \neq b$.

Thm (Paulsen, et.al.)
$p \in C_{qc}^{s}(k,n)$ iff there is a C^*-alg A with a trace τ and PVMs $(e^x)_{x \in [k]}$ in A
(so $\sum_{a \in [n]} e_a^x = 1_A \ \forall x \in [k]$)

projections

s.t. $p(a,b|x,y) = \tau(e^x_a e^y_b)$

<u>Thm</u> (Kim, Paulsen, Schafhauser)

$p \in C^s_{qa}(k,n)$ iff same but the trace must be amenable, i.e. there is a $*$-hom. $\theta : \mathcal{A} \to \mathcal{R}^u$ (with ucp lift $\mathcal{A} \to \ell^\infty(\mathcal{R})$) s.t.

$$\tau = \tau_{\mathcal{R}}^{u} \circ \theta$$

unique trace in $\mathcal{R}^u$

so can assume PVMs live in $\mathcal{R}^u$ and τ is $\tau^u_{\mathcal{R}}$.

If we set $X_n(\mathcal{R}^u) :=$ set of PVMs in $\mathcal{R}^u$ of length n, then

$$s\text{-}val^*(\mathcal{G}) := \sup_{p \in C^s_{qa}(k,n)} val(\mathcal{G}, p)$$

$\tau^u_{\mathcal{R}}$

$$= \underbrace{\sup_{e^1,\dots,e^k \in X_n(\mathcal{R}^{\mathcal{U}})}}_{\text{Not an allowable quantifier.}} \underbrace{\sum \pi(x,y) \sum \tau(e^x_a e^y_b) D(x,y,a,b)}_{\text{atomic formula evaluated in } \mathcal{R}^{\mathcal{U}}}$$

In classical logic, no big deal.

example Groups. $Z(G)$ = center of G.

$\forall x \in Z(G)\ \varphi(x,y)$ is shortcut for

$\forall x (\forall z (xz = zx) \to \varphi(x,y))$

In continuous logic, it is not always the case that, given any formula $\psi(x)$, the "formula" $\sup_{\{x : \psi(x) = 0\}} \varphi(x,y)$ is

equivalent to (really approximable by) an honest formula.

Only possible iff $\forall \varepsilon\, \exists \delta\, \forall a (\Psi(a) < \delta$
$\rightarrow \exists b (\Psi(b) = 0$ and $d(a,b) \leq \varepsilon)$.
"almost -near"

Then call zeroset of Ψ a definable set.

Equivalently, $Z(\Psi^{\mathcal{U}^u}) = Z(\Psi^{\mathcal{U}})^u$.

"Spectral gap and definability"

$X_n(\mathcal{R}^u)$ = set of PVMs in $\mathcal{R}^u$ of length n.

Want $X_n(\mathcal{R}^u)$ to be a definable subset of $((\mathcal{R}^u)_1)^n$.

Magic True and KPS proved it themselves!

Get: s-val*($\mathcal{G}$) is effectively approximable by universal sentences evaluated in $\mathcal{R}$.
Ψ to Halting Problem!

Cor (Gödelian refutation of CEP)
There is no r.e. consistent $T \supseteq T_{II_1}$ s.t. all models of T embed in $\mathcal{R}^{\mathcal{U}}$.

Cor (Many counterexamples to CEP)
There is a sequence $M_0, M_1, \ldots$ of separable II_1 factors, none of which embed in $\mathcal{R}^{\mathcal{U}}$ and s.t. $M_i \not\hookrightarrow M_j^{\mathcal{U}}$ for $i<j$.

Pf Suppose $M_0, \ldots, M_n$ are as in the Cor.
For each $i \leq n$, have universal sentence σ_i s.t. $\sigma_i^{\mathcal{R}} = 0$ but $\sigma_i^{M_i} > 0$.
Take rational $\delta_i \in (0, \sigma_i^{M_i})$ and let

$$T := T_{II_1} \cup \Big\{ \max_{i \leq n} \big(\underbrace{\sigma_i \dot{-} \delta_i}_{\max(\sigma_i - \delta_i, 0)} \big) \Big\} \quad \text{r.e.}$$

By previous Cor, have $M_{n+1} \models T$ s.t. $M_{n+1} \not\hookrightarrow \mathcal{R}^{\mathcal{U}}$.

But $\sigma_i^{M_{n+1}} \le \delta_i < \sigma_i^{M_i}$, so $M_i \not\preceq M_{n+1}^{\mathcal{U}}$. $\boxtimes$

A different kind of game

Model-theoretic forcing, Henkin construction, building models by games (Hodges' book)

C - countable set of constant symbols

2 player game: $\underline{\forall}$belard & $\underline{\exists}$loise

Play finite sets of open conditions of the form $\left|\, \|p(\vec{c})\|_\tau - r \right| < \varepsilon$, $\vec{c}$ finite tuple from C and $p(\vec{x})$ is a $*$-poly.

These conditions must be simultaneously satisfiable in some tracial vNa.

$\forall$ starts.

Each player's play must extend the previous player's play.

Play for ω many rounds.
Assume the play is **definitive**:
for each $p(\vec{x})$, $\vec{c}$, $\|p(\vec{c})\|_\tau$ should be determined after the ω rounds.
$\therefore$ The play of the game determines a unique tracial vNa for which the $*$-subalg. generated by C is dense. **Compiled structure**

Def Given a property P of tracial vNas, say P is **enforceable** if $\exists$ has a strategy so that if she plays according to that strategy, she can always ensure that the compiled algebra has property P.

Conjunction lemma If $(P_n)_{n\in\omega}$ is a list of enforceable properties, so is $\bigwedge_{n\in\omega} P_n$.

trivial center

Example Being a factor is an enforceable property. Let σ be the sentence

$$\sup_x \inf_y \Big(\underbrace{\sqrt{\|x\|_\tau^2 - \tau(x)^2}}_{\theta(x)} \dot{-} \underbrace{\|xy-yx\|_\tau}_{\eta(x,y)} \Big).$$

Farah, Hart, Sherman showed that a tracial vNa M is a factor iff $\sigma^M = 0$.

By Conjunction lemma, it suffices to show for every n, there is m s.t.

$$\theta(c_n) \dot{-} \eta(c_n, c_m) < \varepsilon.$$

$\forall$ plays some set $\Pi(c_0, \ldots, c_k)$. WLOG, $n \le k$.

This play is satisfiable in some tracial vNa M.

But M embeds in a factor $(M \subseteq \overline{M * L(\mathbb{Z})})$.
So $\forall$ play is satisfiable in a factor $N \supseteq M$.
$\therefore \sigma^{N} = 0$. Let c_{n+i}^{N} witness the inf $< \varepsilon$.
$\exists$ returns $\Pi(c_0, \ldots, c_k) \cup \{\theta(c_n) \dot{-} \eta(c_n, c_m) < \varepsilon\}$.

Even enforce a II_1 factor.
ETS you can enforce a projection p
with $\tau(p) = \frac{1}{\pi}$.
$d(c_m, c_m^*) < \varepsilon$, $d(c_m, c_m^2) < \varepsilon$,
$|\tau(c_m) - \frac{1}{\pi}| < \varepsilon$.
Satisfiable b/c every tracial vNa
embeds in a II_1 factor.